Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is. - Paul Erdös

Section A: Induction

Question 1

Prove that $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$, for all $n \geq 1$.

Question 2

(a) Prove that, for all $n \geq 1$, we have:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} + \frac{x^n}{1-x}.$$

(b) Prove that for all $n \geq 1$, $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$.

Question 3

Prove that 5 divides $3^{4n} - 1$, for all $n \geq 1$.

Question 4

Prove that for any **odd** number $m \geq 1$, the number 9 divides $4^m + 5^m$.

Question 5

The *Tower of Hanoi* is a mathematical puzzle that consists of three rods and a number of disks of different sizes, which can slide onto any rod (see Figure 1). The puzzle starts with the disks in a neat stack in ascending order of size on one rod, the smallest at the top, thus making a conical shape. The objective of the puzzle is to move the entire stack to another rod. The player is only allowed to move one disk at a time, and only smaller disks can be on top of a bigger disk.

Find and prove a formula for the least number of moves required to move a Tower of Hanoi with n disks to another rod. (Hint: find the least number of moves for $n = 1$, $n = 2$, and $n = 3$, then conjecture a formula, and prove your formula using induction.)

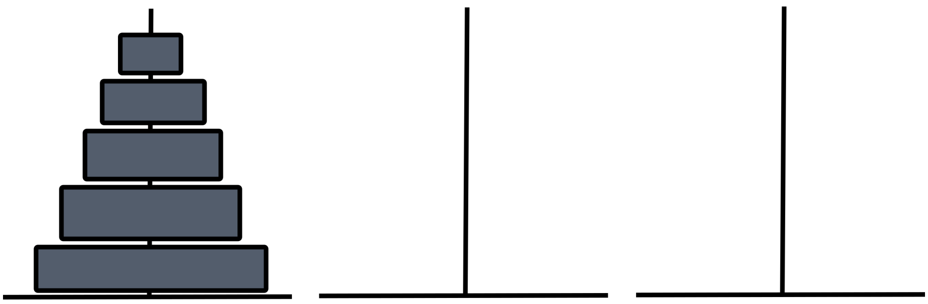

Figure 1: The *Tower of Hanoi* puzzle, with $n = 5$ disks.

Question 6

(a) Show that $n! \leq n^n$ for all $n > 0$.

(b) $(n+1)^{(n-1)} \leq n^n$ for all $n > 0$.

Question 7

What is wrong with the following proof? **Theorem**. All babies have the same color eyes. **"Proof"**. The base case is clear: one baby has the same color eyes as herself or himself. Now suppose we have $n + 1$ babies, and name them $\{B_1, \ldots, B_n, B_{n+1}\}$. By the induction hypothesis, the babies in sets $\{B_1, \ldots, B_n\}$ and those in $\{B_2, \ldots, B_{n+1}\}$ have the same color eyes, and hence all of the babies $B_1, \ldots, B_{n+1}$ have the same color eyes. $\square$

Section B: Complete Induction

Question 8

Prove that any natural number $n \geq 2$ either is a prime or factors into a product of primes.

Question 9

Prove that the sum of the interior angles of an n-sided convex polygon is $180 \cdot (n-2)$ degrees.