Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is. - Paul Erdös

Section A: Induction

Question 1

Prove that $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$, for all $n \geq 1$.

Solution:

We use induction. The base case is $n = 1$ which is obvious:

$$1 = 1^3 = \left(\frac{1 \cdot 2}{2}\right)^2 = 1^2 = 1.$$

Now, let us prove the induction step. For this, assume that the result is true for some $n \geq 1$ and we need to prove that the result is true for $n+1$. Hence, we need to prove:

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 \ \stackrel{?}{=} \ \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

This is indeed true because:

$$\begin{aligned} 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2 \cdot \left(\frac{n^2}{4} + (n+1)\right) \\ &= (n+1)^2 \cdot \left(\frac{n^2+4n+4}{4}\right) \\ &= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2. \end{aligned}$$

This proves the induction step. Hence, by the Principle of Mathematical Induction, the equality is true for all $n \geq 1$.

Question 2

(a) Prove that, for all $n \geq 1$, we have:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} + \frac{x^n}{1-x}.$$

(b) Prove that for all $n \geq 1$, $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$.

Solution:

(a). We will use induction. The base case ($n = 1$) is "trivial" (= almost obvious):

$$1 + \frac{x}{1-x} = \frac{1-x+x}{1-x} = \frac{1}{1-x}.$$

Now we prove the induction step. Suppose the equality is true for some $n \geq 1$ and we want to prove it for $n+1$. Since we can assume the equality for n, we also know that:

$$\frac{1}{1-x} - \frac{x^n}{1-x} = 1 + x + x^2 + \cdots + x^{n-1}.$$

We will use the above equation below. At this point, it remains to verify the following equality:

$$\frac{1}{1-x} \stackrel{?}{=} 1 + x + x^2 + \cdots + x^{n-1} + x^n + \frac{x^{n+1}}{1-x}.$$

Let's start working with the right hand side:

$$\begin{aligned} 1 + x + x^2 + \cdots + x^{n-1} + x^n + \frac{x^{n+1}}{1-x} &= \frac{1}{1-x} - \frac{x^n}{1-x} + x^n + \frac{x^{n+1}}{1-x} \\ &= \frac{1 - x^n + x^n(1-x) + x^{n+1}}{1-x} \\ &= \frac{1 - x^n + x^n - x^{n+1} + x^{n+1}}{1-x} = \frac{1}{1-x} \end{aligned}$$

as desired. Hence, the induction step is proven, and by the Principle of Mathematical Induction, the equality is true for all $n \geq 1$.

(b). Now, plug $x = 2$ in the equality we just prove to get:

$$\frac{1}{1-2} = 1 + 2 + 2^2 + \cdots + 2^{n-1} + \frac{2^n}{1-2}$$

or

$$2^n - 1 = 1 + 2 + 2^2 + \cdots + 2^{n-1}$$

as needed.

Question 3

Prove that 5 divides $3^{4n} - 1$, for all $n \geq 1$.

Solution:

We use Induction. The base case, $n = 1$ is clear since $3^4 - 1 = 80 = 5 \cdot 16$ is divisible by 5. Now suppose that this property is true for some $n \geq 1$ and so, $3^{4n} - 1$ is divisible by 5. Let us write then $3^{4n} - 1 = 5k$ for some natural number $k \geq 1$. We need to prove that $3^{4(n+1)} - 1$ is also divisible by 5. But:

$$3^{4n+4} - 1 = 3^4(3^{4n}) - 1 = 81(5k+1) - 1 = 405k + 81 - 1 = 405k + 80 = 5 \cdot (81k + 16).$$

Hence, $3^{4(n+1)} - 1$ is also divisible by 5. Hence, the induction step is proven, and by the Principle of Mathematical Induction, the property is true for all $n \geq 1$.

Question 4

Prove that for any **odd** number $m \geq 1$, the number 9 divides $4^m + 5^m$.

Solution:

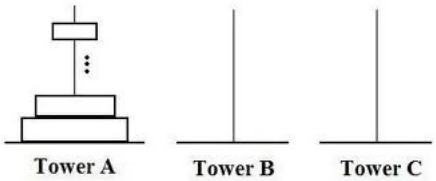
Every odd number m can be written as $2n+1$, for some $n \geq 0$ (by the division theorem the remainder when dividing by 2 is 1 or 0, and since m is odd, it must be a remainder of 1). We use induction on n to prove the result. The base case is trivial since $4+5=9$ is divisible by 9. Suppose next that $n \geq 1$ is a number such that $4^{2n+1} + 5^{2n+1}$ is divisible by 9. Let us write $4^{2n+1} + 5^{2n+1} = 9k$ for some $k \geq 1$. We shall show that $4^{2(n+1)+1} + 5^{2(n+1)+1}$ is also divisible by 9. Indeed:

$$\begin{aligned} 4^{2n+3} + 5^{2n+3} &= 16 \cdot 4^{2n+1} + 25 \cdot 5^{2n+1} = 16(4^{2n+1} + 5^{2n+1}) + 9 \cdot 5^{2n+1} \\ &= 16 \cdot 9k + 9 \cdot 5^{2n+1} = 9(16k + 5^{2n+1}). \end{aligned}$$

Hence, $4^{2(n+1)+1} + 5^{2(n+1)+1}$ is also divisible by 9. Hence, the induction step is proven, and by the Principle of Mathematical Induction, the property is true for all numbers $n \geq 1$, and so $4^m + 5^m$ is divisible by 9 for all odd numbers $m \geq 1$. **Notice that this property is not true for even m, e.g.** $4^2 + 5^2 = 16 + 25 = 41$.

Question 5

Prove a formula for the least number of moves required to move a Tower of Hanoi with n disks to another pole.



Solution:

We shall prove using induction that the least number of moves is $2^n - 1$, for all $n \geq 1$. The base case is trivial: if you only have one disk, then it only takes one move to take one disk to another pole.

Now, suppose that the least number of moves to take a tower of n disks to another pole is exactly $2^n - 1$. Let us suppose that we have a tower of $n+1$ disks. In order to move the tower, we will eventually have to move the last disk at the bottom of the pile. In order to access the last disk, we first need to move the n disks on top to another pole. Notice that we need them all in one single pole to allow space to move the largest disk to another pole. This will take at least $2^n - 1$ moves, by the induction hypothesis. Then we can move the $(n+1)$th disk to another pole (at least 1 extra move) and, last, we need to move the other n disks back on top of the largest disk. This last procedure is identical to moving a tower of n disks to another pole, hence, by the induction hypothesis, it will take at least $2^n - 1$ moves. In total, we have used at least the following number of moves:

$$(2^n - 1) + 1 + (2^n - 1) = 2 \cdot 2^n - 1 = 2^{n+1} - 1.$$

Hence, the induction step is proven, and by the Principle of Mathematical Induction, the property is true for all $n \geq 1$.

Question 6

(a) Show that $n! \leq n^n$ for all $n > 0$.

(b) $(n+1)^{(n-1)} \leq n^n$ for all $n > 0$.

Solution:

(a). For $n = 1$ we have $1! = 1 \leq 1^1 = 1$. Now suppose that $n! \leq n^n$. Then:

$$(n+1)! = (n+1) \cdot n! \leq (n+1)n^n \leq (n+1)(n+1)^n \leq (n+1)^{(n+1)},$$

since $n < (n+1)$. Therefore, by the principle of mathematical induction, the result is true for all $n \geq 1$.

(b). Let us start with the base case $n = 1$: $1 = 2^0 \leq 1^1 = 1$. Now, let us assume that $(n+1)^{(n-1)} \leq n^n$ for some $n \geq 1$. Equivalently, we may assume that:

$$\left(\frac{n+1}{n}\right)^n \leq n+1.$$

We shall prove that:

$$\left(\frac{n+2}{n+1}\right)^{n+1} \stackrel{?}{\leq} n+2.$$

Indeed, notice that $\frac{n+2}{n+1} \leq \frac{n+1}{n}$, for all $n \geq 1$ and therefore:

$$\begin{aligned}\left(\frac{n+2}{n+1}\right)^{n+1} &= \left(\frac{n+2}{n+1}\right)^n \cdot \left(\frac{n+2}{n+1}\right) \leq \left(\frac{n+1}{n}\right)^n \cdot \left(\frac{n+2}{n+1}\right) \\ &\leq (n+1) \cdot \frac{(n+2)}{(n+1)} \leq n+2.\end{aligned}$$

Therefore, by the principle of mathematical induction, the result is true for all $n \geq 1$.

Note: Have you noticed how HARD it is to prove, using Calculus, that the function $F(x) = (x+1)^{(x-1)}$ is less than $G(x) = x^x$? However, induction makes it easy!

Question 7

What is wrong with the following proof? **Theorem**. All babies have the same color eyes. **"Proof"**. The base case is clear: one baby has the same color eyes as helself or himself. Now suppose we have $n+1$ babies, and name them $\{B_1, \ldots, B_n, B_{n+1}\}$. By the induction hypothesis, the babies in sets $\{B_1, \ldots, B_n\}$ and those in $\{B_2, \ldots, B_{n+1}\}$ have the same color eyes, and hence all of the babies $B_1, \ldots, B_{n+1}$ have the same color eyes. $\square$

Solution:

The problem is at the base case. The theorem can be restated as "Any subset of n babies have the same color eyes". Since the statement is a comparison among babies, the base case should be the case $n = 2$, and not the case $n = 1$. In other words, to use the induction step mentioned in the "proof", one would have to first prove the case $n = 2$: any two babies have the same color eyes. But, of course, this is already false.

Section B: Complete Induction

Question 8

Prove that any natural number $n \geq 2$ either is a prime or factors into a product of primes.

Solution:

We shall prove this using complete induction. The base case $n = 2$ is clear, since 2 is prime. Now suppose that all integers $2 \leq k < n$ are either a prime or factor into a product of primes. If n is prime, then we are done. Otherwise, if n is not prime, then it is composite. Thus there exist positive integers a, b with $1 < a, b < n$ and $n = a \cdot b$. By the induction hypothesis, both a and b are either primes or a product of primes, say

$$a = p_1 \cdot p_2 \cdots p_r, \quad b = q_1 \cdot q_2 \cdots q_s$$

for some primes $p_1, \ldots, p_r, q_1, \ldots, q_s$. Hence

$$n = ab = p_1 \cdot p_2 \cdots p_r \cdot q_1 \cdot q_2 \cdots q_s$$

is a product of primes. Hence, the induction step is proven, and by the Strong Principle of Mathematical Induction, the property is true for all $n \geq 2$.

Question 9

Prove that the sum of the interior angles of an n-sided convex polygon is $180(n-2)$ degrees.

Solution:

We shall prove this using complete induction, for all $n \geq 3$. We will assume the well known fact that the sum of the interior angles of a triangle is 180 degrees (can you find a proof of this well known fact?). This shows the base case. Now, assume that for every $3 \leq k < n$, the sum of the interior angles of every k-sided convex polygon is $180(k-2)$ degrees. Let P be an n-sided convex polygon, for some $n > 3$. By drawing a segment between two non-consecutive vertices, we may divide the polygon P into two polygons Q and R which share an edge. Notice that both Q and R are convex polygons, and they have strictly less than n sides each. Let's say Q and R have q and r sides, respectively. Also notice that $(q-1)+(r-1) = n$ and the sum of the interior angles of P is equal to the sum of the interior angles of Q plus the sum of the interior angles of R. By the induction hypothesis, the sum of the interior angles of Q and R are $180(q-2)$ and $180(r-2)$ respectively. Therefore, the sum of the interior angles of P is:

$$180(q-2) + 180(r-2) = 180(q+r-4) = 180(n-2).$$

Hence, the induction step is proven, and by the Principle of Mathematical Induction, the property is true for all $n \geq 3$.