God made the integers, all else is the work of man. (Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.). - Leopold Kronecker

Question 1. Prove that there are infinitely many primes of the form $4n - 1$.

Question 2. Prove that there are infinitely many primes of the form $6n - 1$.

Question 3. Let $a_1 = 2$ and $a_{n+1} = a_n(a_n - 1) + 1$. First prove that $a_{n+1} = a_1 a_2 \cdots a_n + 1$. Then, prove that for all $m \neq n$, the numbers a_m and a_n are relatively prime.

Question 4. Prove that for any $n \geq 1$ there are n consecutive composite numbers.

Question 5. Prove that for any $n \geq 2$ there is a prime p with $n < p \leq n! + 1$.

Question 6. Find the least non-negative residues of (a) $365 \bmod 5$, (b) $-3122 \bmod 3$, (c) $3122082546 \bmod 10$, and (d) $-2445678 \bmod 10$.

Question 7. Find one integer $a \in \mathbb{Z}$ that satisfies, simultaneously, both congruences $a \equiv 5 \mod 8$ and $a \equiv 3 \mod 7$.

Question 8. Show that if $n > 4$ is not prime then $(n-1)! \equiv 0 \mod n$.

Question 9. Prove the following properties of congruences:

(a) If $a \equiv b \mod n$ then $ka \equiv kb \mod n$.

(b) If $a \equiv b \mod n$ and $a' \equiv b' \mod n$ then $a + a' \equiv b + b' \mod n$.

Question 10. Use congruences to show that $6 \cdot 4^n \equiv 6 \mod 9$ for any $n \geq 0$.

Question 11. Find the least nonnegative residues.

(a) $5^{18} \bmod 7$.

(b) $68^{105} \bmod 13$.

(c) $6^{47} \bmod 12$.

Question 12. Show that $5^e + 6^e \equiv 0 \mod 11$ for all odd numbers e.

Question 13. Prove the part (a), then find the least nonnegative residue modulo 7, 11 and 13 in parts (b), (c) and (d).

(a) A number N is congruent modulo 7, 11, or 13, to the alternating sum of its digits in base 1000. (For example, $123456789 \equiv 789 - 456 + 123 \equiv 456 \bmod 7, 11, \text{ or } 13$.)

(b) 11233456,

(c) 58473625,

(d) $100,000,000,000,000,001$.

Question 14. Find divisibility tests for numbers in base 34 for $2, 3, 5, 7, 11$ and 17.

Question 15. Show that $2^{560} \equiv 1 \mod 561$.