If the Sun refused to shine,
I don't mind, I don't mind.
If the mountains fell in the sea,
Let it be, it ain't me.
Now, if six turned out to be nine,
Oh I don't mind, I don't mind...
Jimi Hendrix, "If Six Was Nine," from the album *Axis: Bold as Love*, 1967.

Question 1. If six turned out to be nine...
(a) ... that is, if $6 \equiv 9 \bmod m$, what would the value of $m > 1$ be?

(b) Now, if $6 \equiv 69 \bmod m$, what are the possible values for $m > 1$?

Question 2. Find the smallest number ≥ 120120 which is divisible by no prime $p < 20$, using congruences. (Hint: calculate $120120 \bmod p$, for every prime $p < 20$.)

Question 3. Find all $x \in \mathbb{Z}$ that satisfy the following linear congruence, or explain why no integral solutions exist (these are individual congruences, and not a system!).
(a) $6x \equiv 9 \bmod 11$,

(b) $6x \equiv 11 \bmod 9$,

(c) $6x \equiv 9 \bmod 15$.

Question 4. Solve the following systems:

$$\begin{cases} x \equiv 2 \bmod 7 \\ x \equiv 4 \bmod 8 \\ x \equiv 3 \bmod 9 \end{cases}, \quad \begin{cases} z \equiv 5 \bmod 7 \\ z \equiv 2 \bmod 8 \\ z \equiv 1 \bmod 9 \end{cases}, \quad \begin{cases} y \equiv 1 \bmod 7 \\ y \equiv 3 \bmod 8 \\ y \equiv 6 \bmod 9 \end{cases}.$$

Question 5. Solve the following systems:

$$\begin{cases} x \equiv -3 \bmod 11 \\ x \equiv 103 \bmod 13 \\ x \equiv 3 \bmod 15 \end{cases}, \quad \begin{cases} y \equiv 25 \bmod 11 \\ y \equiv 35 \bmod 13 \\ y \equiv 31 \bmod 15 \end{cases}.$$

Question 6. Solve:

$$\begin{cases} x \equiv 1 \quad \bmod 2 \\ x \equiv 2 \quad \bmod 5 \\ x \equiv 5 \quad \bmod 6 \\ x \equiv 5 \quad \bmod 12. \end{cases}$$

Question 7. A prime p is a safe prime if $p = 2q + 1$ where q is also prime. The prime q, in turn, is called a Sophie Germain prime. For instance, $p = 5 = 2 \cdot 2 + 1$ and $p = 7 = 2 \cdot 3 + 1$ are the first two safe primes, and $q = 2$ and $q = 3$ are the first two Sophie Germain primes. Suppose that $p > 7$ is a safe prime and prove the following.
(a) Show that $p \equiv 2 \bmod 3$.

(b) Show that $p \equiv 3 \bmod 4$.

(c) Show that if $p > 11$, then $p \not\equiv 1 \bmod 5$.

(d) Use the previous congruences to show that $p \equiv 23, 47$ or $59 \bmod 60$.

(e) Use (d) to find 10 safe primes larger than 1000.

Question 8.

(a) Find all solutions for the congruence $x^2 \equiv 1 \bmod 8$.

(b) Find all solutions for $x^2 \equiv 1 \bmod 5$.

(c) Use (a) and (b) and the Chinese remainder theorem to find all solutions for $x^2 \equiv 1 \mod 40$.

Question 9.

(a) Find all the congruence classes modulo 35 that are zero-divisors in $\mathbb{Z}/35\mathbb{Z}$.

(b) Find all the congruence classes modulo 35 that are units in $\mathbb{Z}/35\mathbb{Z}$.

(c) For each unit modulo 35, find its multiplicative inverse.

(d) Repeat parts (a), (b) and (c) for the ring $\mathbb{Z}/11\mathbb{Z}$.

Question 10.

(a) Justify the following congruence modulo 11:

$$\begin{aligned} 10! &\equiv 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \\ &\equiv 1 \cdot 2 \cdot 2^{-1} \cdot 3 \cdot 3^{-1} \cdot 5 \cdot 5^{-1} \cdot 7 \cdot 7^{-1} \cdot 10 \\ &\equiv 1 \cdot 10 \equiv -1 \bmod 11. \end{aligned}$$

(b) Generalize the formula in (a) to prove that if p is any prime then $(p-1)! \equiv -1 \bmod p$.